STUDY OF THE EFFECTIVE STABILITY IN THE SPATIAL RESTRICTED PROBLEM OF THREE BODIES

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ABSTRACT: We study the spatial restricted problem of three bodies in the light of Nekhoroshev theory of stability over large time intervals. We consider in particular the Sun-Jupiter model and the Trojan asteroids in the neighborhood of the Lagrangian point L_4. We find a region of effective stability around the point L_4 such that if the initial point of an orbit is inside this region the orbit is confined in a slightly larger neighborhood of the equilibrium (in phase space) for a very long time interval. By comparing this result to the one obtained for the planar problem we see that the regions of stability in the two cases are of the same magnitude.

1. Introduction

The study of a Hamiltonian system in the neighborhood of an elliptic equilibrium point is of interest in many fields of mathematical physics and astronomy. Let us consider an analytic Hamiltonian H with n degrees of freedom, having an elliptic equilibrium point at the origin. Rigorous results proving the existence of orbits which do not leave a neighborhood of the equilibrium can be given in the framework of KAM theory, under generic conditions of non-resonance and non-degeneracy. KAM theorem guarantees the existence of many n-dimensional invariant tori around the origin. However, such invariant tori do not fill an open region, i.e. the possibility of the so-called Arnold diffusion cannot be excluded, except for the two dimensional case.

An alternative approach is to look for results that are valid over a finite time interval but give an effective bound on the Arnold diffusion. This goal can be achieved by constructing the normal form of the Hamiltonian around the origin. Normal forms are a standard tool in Celestial Mechanics for studying the dynamics in the neighborhood of an elliptic equilibrium point. Usually these normal forms are obtained as divergent series but their asymptotic character makes them useful. Roughly speaking one shows that the system admits a number of approximate integrals the variation of which in time can be controlled to be small for an extremely long time. In these cases we have effective stability, i.e. even when an orbit is not stable, the time needed for it to leave the neighborhood of the equilibrium is larger than the expected lifetime of the studied physical system. This is the basis to derive the classical Nekhoroshev's estimates (Nekhoroshev 1977).

A scientific field where the Nekhoroshev theory has been applied is the problem of the stability of the Trojan asteroids. Simple models for describing the motion of an asteroid, are the two dimensional (2D) planar, and the three dimensional (3D) spatial restricted three body problem. According to Nekhoroshev theory, one has to estimate the rate of diffusion around the elliptic equilibrium Lagrangian point L_4. The problem has been previously investigated by Giorgilli et al. (1989), Celletti and Giorgilli (1991), Skokos et al. (1996), Skokos (1997) and Giorgilli and Skokos (1997) (hereafter paper I).

The estimation of the region of effective stability by Giorgilli et al. (1989) and Celletti and Giorgilli (1991) was realistic but the region where the real asteroids were actually found was larger by a factor 300 (in the best case) to 3,000 compared to the estimated stability region. This estimation was significantly improved in paper I, since the region found in the planar restricted three body problem was big enough to include 4 real asteroids, while most of them fail to be inside this region by a factor 10. This result underlined the fact that Nekhoroshev theory can give meaningful estimates, applicable to real systems. On the other hand Arnold diffusion, which, as already mentioned, can drive some orbits with initial conditions near the equilibrium point L_4 to regions of the phase space far away from it, appears only in the spatial case and not in the planar one.

In the present paper we study the spatial case following a similar procedure to the one used in paper I. We numerically compute the normal form up to order 30, which is really a hard task to do since one has to manipulate functions with a huge number of coefficients. Some changes in the scheme used in paper I improve slightly the estimation of the size of the stability region. In particular the expansion of the Hamiltonian of the system in a power series suitable for the application of the normal form scheme is computed with higher accuracy than before. Also a more accurate calculation of the time needed for an orbit to leave a particular region of the phase space around the point L_4 (escape time) is provided.

2. The Hamiltonian and the normal form of the system

The spatial restricted problem of three bodies, in particular for the Sun (S), Jupiter (J) and asteroid (A) case can be described as follows: we study the motion of an asteroid A of infinitesimal mass, orbiting in the gravitational field of two primaries S and J with masses equal to $1-\mu$ and μ respectively, which are assumed to revolve in circular orbits around their common center of mass.

The formalism we use is similar to the one used in paper I, generalized for three degrees of freedom. For the sake of completeness we recall the main points of this formalism. We introduce a uniformly rotating frame (O, q_1, q_2, q_3) so that its origin is located at the center of mass of the Sun-Jupiter system, with the Sun always at the point $(\mu, 0, 0)$ and Jupiter at the point $(\mu-1, 0, 0)$. The physical units are chosen so that the distance between Jupiter and the Sun is 1, $\mu=9.5387536\cdot10^{-4}$ and the angular velocity of Jupiter is 1. The time unit is $(2\pi)^{-1}T_J$ where T_J is the period of the circular motion of Jupiter around the Sun. Thus the age of the universe is about 10^{10} time units. The Hamiltonian of the system is:

$$H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+q_2p_1-q_1p_2-\frac{1-\mu}{\sqrt{(q_1-\mu)^2+q_2^2+q_3^2}}-\frac{\mu}{\sqrt{(q_1+1-\mu)^2+q_2^2+q_3^2}} \quad (1).$$

In order to bring the Hamiltonian in a form suitable for the application of the normal form scheme we perform a sequence of transformations.

a) We introduce a uniformly rotating frame with its origin on the Sun (S) using the generating function $W_3=-(Q_1+\mu)\,p_1-Q_2\,p_2-Q_3\,p_3+\mu\,Q_2$, where $Q_1, Q_2, Q_3, P_1, P_2, P_3$ are the heliocentric coordinates.

b) We introduce cylindrical coordinates P, Θ, Z, via the canonical transformation generated by $W_3=-P(P_1\cos\Theta+P_2\sin\Theta)-Z\,P_3$.

c) We move the origin of the coordinate system to the point L_4 by using the canonical transformation generated by $W_2=p_x\,(P-1)+(p_y+1)\,\Theta-2\pi\,p_y/3+p_z\,Z$. Then the Hamiltonian becomes:

$$H=\frac{1}{2}\left[p_x^2+\frac{(p_y+1)^2}{(x+1)^2}+p_z^2\right]-p_y-\mu(x+1)\cos\left(y+\frac{2\pi}{3}\right)-\frac{1-\mu}{\sqrt{(x+1)^2+z^2}}-$$
$$-\frac{\mu}{\sqrt{(x+1)^2+z^2+1+2\,(x+1)\cos\left(y+\frac{2\pi}{3}\right)}} \qquad (2),$$

where x, y, z, p_x, p_y, p_z are the new canonical coordinates.

d) We expand the Hamiltonian (2) in Taylor series around the point L_4 (x = y = z = px = py = pz = 0) using the computer algebra platform "Mathematica" (Wolfram Research Inc.). The program allows us to compute the coefficients of the expansion using arbitrary precision arithmetics, while in paper I the corresponding expansion was made with less accuracy. This change improves the credibility of our computation.

e) Following paper I, we introduce a canonical transformation which brings the quadratic part of the Hamiltonian to the diagonal form $H_2=\sum_{j=1}^{3}\omega_j\cdot(x_j^2+y_j^2)/2$, where x_1, x_2, x_3, y_1, y_2, y_3 are the canonical coordinates and $\omega_1=9.967575\cdot10^{-1}$, $\omega_2=-8.046388\cdot10^{-2}$, $\omega_3=1$ are the frequencies. After this transformation the Hamiltonian of the system is brought to the form $H=\sum_{s\geq2}H_s$, where H_s is a homogeneous polynomial of degree s.

f) Finally, following Giorgilli et al. (1989), we construct the normal form $Z^{(r)}$ up to order r:

$$Z^{(r)}(x_1',x_2',x_3',y_1',y_2',y_3')=Z_2+Z_3+Z_4+\ldots+Z_r+Y^{(r)} \qquad (3),$$

where Z_s , s=2,...,r is a homogeneous polynomial of degree s in the new "normal variables" $x_1', x_2', x_3', y_1', y_2', y_3'$. The term "normal form" means, that Z is a function of the quantities $I_j'=\frac{1}{2}(x_j'^2+y_j'^2)$, j=1,2,3, so that the system is formally integrable. $Y^{(r)}$ is a remainder, actually a power series starting with terms of degree r+1. The algorithm for the computation of the normal form is described by Giorgilli (1979).

In paper I, where the planar problem (2D case) was studied, the power series of 4 variables were truncated at order $\tilde{r}=35$. A function of 4 variables expanded up to order 35 requires 82,251 coefficients, while the process of constructing the normal form requires the computation of several functions with a total of 2,549,782 coefficients. In the spatial problem (3D case) we use expansions of functions of 6 variables up to order $\tilde{r}=30$. This is a much harder task compared to the 2D case since a function of 6 variables expanded up to order 30 requires 1,947,792 coefficients and the program which calculates the normal form manipulates 55,929,459 coefficients.

3. Estimation of the size of the effective stability region

The normal form $Z^{(r)}$ admits three approximate first integrals of the form:

$$I_j'=\frac{1}{2}(x_j'^2+y_j'^2)\quad,\quad j=1,2,3 \qquad (4).$$

The variation of these quantities in time, is given by:

$$\dot{I}'_j = \left[I'_j ; Z^{(r)} \right] \overset{(3)}{=} \left[I'_j , Y^{(r)} \right], j=1,2,3 \quad (5),$$

which is a power series starting with terms of degree r+1. We remark that [f , g] denotes the Poisson bracket of functions f and g.

We introduce now suitable domains in the phase space where we study the stability properties of the system and also a norm in these domains, in order to estimate the time variations of the three approximate integrals (4). For arbitrary fixed positive constants R_1, R_2, R_3 we consider a family of domains of the form:

$$\Delta_{\rho R} = \left\{ (x', y') \in R^6 : x_j'^2 + y_j'^2 \le \rho^2 R_j^2 ,\ j=1,2,3 \right\} \quad (6),$$

where ρ is a positive parameter and X' stands for x'_1, x'_2, x'_3 and y' for y'_1, y'_2, y'_3. For $(x', y') \in \Delta_{\rho R}$ we have $I'_j \le \rho^2 R_j^2/2$ for j=1,2,3. The norm $\|f\|_{\rho R}$ of a homogeneous polynomial $f(x',y')$ of degree s in the domain $\Delta_{\rho R}$ does not exceed the quantity (Skokos, 1997):

$$\|f\|_{\rho R} \le \frac{\rho^s}{2^{s/2}} \sum_{j_1 j_2 j_3 k_1 k_2 k_3} | C_{j_1 j_2 j_3 k_1 k_2 k_3} | \, R_1^{j_1+k_1} R_2^{j_2+k_2} R_3^{j_3+k_3} \quad (7),$$

where $C_{j_1 j_2 j_3 k_1 k_2 k_3}$ are the complex coefficients of $f(x',y')$ when f is transformed in complex variables ξ, η via the relations $x'_j = (\xi_j + i\,\eta_j)/\sqrt{2}$ and $y_j = i\,(\xi_j - i\,\eta_j)/\sqrt{2}$.

We assume that the initial point of an orbit lies in the domain $\Delta_{\rho_0 R}$ for some positive ρ_0, and we ask the orbit to be confined inside a domain $\Delta_{\rho R}$ with $\rho_0<\rho$ for a finite time interval τ (escape time). Since $\dot{I}'_j = dI_j / dt$, we get

$$dt \ge \frac{dI'_j}{\sup_{\Delta_{\rho R}} |\dot{I}'_j|}, j = 1, 2, 3 \quad (8),$$

where $\sup_{\Delta_{\rho R}} |\dot{I}'_j|$ denotes the supremum norm of $\dot{I}'_j$, over the domain $\Delta_{\rho R}$. As explained in paper I, assuming that ρ is smaller than the half of the convergence radius of the remainder we get:

$$\sup_{\Delta_{\rho R}} |\dot{I}'_j| < 2 \cdot \left\| \left[I'_j , Y^{(r)}_{r+1} \right] \right\|_{\rho R} = 2\rho^{r+1} \cdot \left\| \left[I'_j , Y^{(r)}_{r+1} \right] \right\|_{R} , \quad j=1,2,3 \quad (9),$$

where $Y^{(r)}_{r+1}$ is the first term of the remainder, which is a homogeneous polynomial of order r+1 and can be easily computed. By integrating both parts of Eq. (8) and using Eq. (9), we estimate the minimum escape time as:

$$\tau_r(\rho_0, \rho) = \min_{j=1,2,3} \frac{R_j^2}{2(r-1)\left\|\left[I'_j, Y_{r+1}^{(r)}\right]\right\|_R} \left[\frac{1}{\rho_0^{r-1}} - \frac{1}{\rho^{r-1}}\right] \quad (10).$$

In order to have the minimum escape time as a function of ρ_0 we eliminate the dependence of $\tau_r(\rho_0, \rho)$ on ρ and r. It is evident that for a given order r the escape time goes to infinity as the radius of the outer domain grows. So by fixing ρ to be equal to $\lambda\rho_0$, with $\lambda > 1$ we get:

$$\tau_{r,\lambda}(\rho_0) = \min_{j=1,2,3} \frac{R_j^2}{2(r-1)\rho_0^{r-1}\left\|\left[I'_j, Y_{r+1}^{(r)}\right]\right\|_R} \left[1 - \frac{1}{\lambda^{r-1}}\right] \quad (11).$$

By giving a fixed value to λ we eliminate the dependence of the minimum escape time on the radius of the final domain. In particular, we put λ=1.2, which means that the radius of the final domain is 20% greater than the radius of the initial domain. Then we compute $\tau_{r,1.2}(\rho_0)$ via Eq. (11) for r running from 3 to the maximum order $\tilde{r}-1$, for every value of ρ_0. We choose the optimal order r_{opt} of the expansion as the one that gives the maximum value of the escape time. Thus we get the maximum escape time T as function of only the radius ρ_0 of the initial domain.

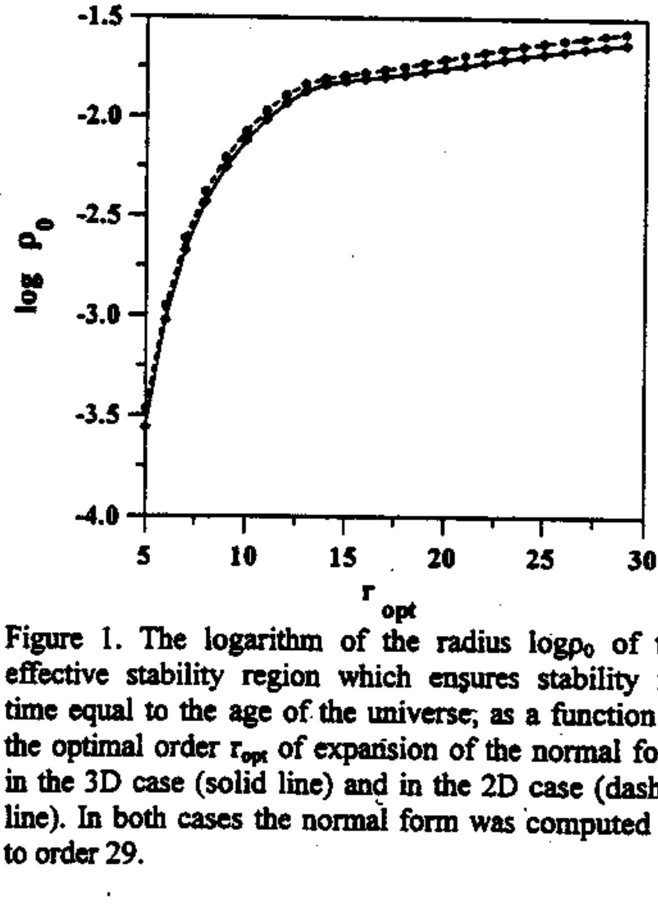

Figure 1. The logarithm of the radius $\log\rho_0$ of the effective stability region which ensures stability for time equal to the age of the universe, as a function of the optimal order r_{opt} of expansion of the normal form in the 3D case (solid line) and in the 2D case (dashed line). In both cases the normal form was computed up to order 29.

For a general discussion and for making the results comparable to the ones in paper I we put $R_1 = R_2 = R_3 = 1$. A meaningful time interval for our system is the estimated age of the universe, which in our time units is 10^{10}. The value of the radius ρ_0 of the initial domain is found to be $\rho_0 \approx 2.371 \cdot 10^{-2}$ in the 3D case and $\rho_0 \approx 2.692 \cdot 10^{-2}$ in the 2D case. In both cases the optimal order of the expansion is $r_{opt} = 29$.

The best previously found estimation of the radius of the effective stability region, was obtained in paper I in the 2D case, namely $\rho_0 \approx 2.911 \cdot 10^{-2}$, where the normal form was computed up to order 34. In that case the optimal order was $r_{opt} = 34$. The radius of the effective stability region in the spatial case is about 12% smaller than the one computed for the planar case for $r_{opt} = 29$ and about 19% smaller than the one found in paper I for $r_{opt} = 34$. So, the estimated stability region in the 3D case is a realistic one since it is comparable to the region found in Paper I, which was big enough to include four real asteroids. This result improves significantly previous estimations of the effective stability region in the spatial restricted three body problem (Giorgilli et al., 1989 and Celletti and Giorgilli, 1991).

The estimated radii in the 3D and the 2D cases, for the same order of expansion of the normal form, are relatively close to each other since they are of the same order of magnitude, with the radius computed in the 3D case being always smaller, as seen in Figure 1. Thus the Arnold diffusion, which appears only in the 3D case, does not affect the size of the effective stability region significantly.

We notice in Figure 1 that up to order $r_{opt} = 14$ the increment of the order r_{opt} improves the estimation of the radius significantly both in the 3D and the 2D case. For orders greater than 15 the increment of the order leads to big increment of the computational effort needed for the construction of the normal form (mainly in the 3D case), but to relatively small improvements for the estimated radii. So, it is evident that pushing the computation of the normal form to higher orders becomes impractical for the 3D case.

4. Conclusions

We studied the spatial circular restricted problem of three bodies in the spirit of Nekhoroshev theory, considering in particular the problem of the stability of the Trojan asteroids around the Lagrangian point L_4 in the Sun-Jupiter-Asteroid model.

The estimated size of the effective stability region in the 3D case is comparable to the size of the region found in paper I for the planar problem, which includes 4 real asteroids. This result significantly improves older estimations of the effective stability region in the spatial restricted three body problem.

The radii of the effective stability region in the general spatial and planer cases are close to each other for the same order of expansion of the normal form, with the radius computed for the spatial case being always slightly smaller. Thus, Arnold diffusion does not affect the size of the effective stability region significantly.

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